Pergamon

Int. J. Solids Structures Vol. 32, No. 13, pp. 1907–1925, 1995

0020-7683/95 \$9.50 + .00

0020–7683(94)00238–X

INCREMENTAL STRESSES IN LOADED ORTHOTROPIC CIRCULAR MEMBRANE TUBES—I. THEORY

AVINOAM LIBAI and DAN GIVOLI
Department of Aerospace Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel

(*Received* 11 *April* 1994; *in revised form* 26 *August* 1994)

Abstract—Starting from the equations of general elastic nonlinear membrane theory in intrinsic form, the equations governing the incremental state of stress in an orthotropic circular membrane tube are derived and discussed. The tube is initially subjected to uniform internal pressure and to longitudinal extension, which lead to large homogeneous deformation. Then some changes in loading and/or geometry are considered, e.g. an additional load is applied, the shape of the boundary is changed or a slit is formed in the membrane. These changes are regarded as small perturbations on the initial homogeneous state of stress. The general form as well as some simplified forms of the equations are presented, leading finally to a set of two equations in terms of two scalar potential functions. Two different variational formulations for the problem are also presented, each of which may serve as a basis for numerical treatment.

1. INTRODUCTION

The field equations of the membrane theory of shells are formally derived by deleting the stress couples from the equations of equilibrium. As a consequence, the transverse shear stress resultants also vanish, and only the in-plane stress resultants (so-called membrane forces) may assume nonzero values. There are two types of membranes: those which serve as an "interior" approximation to the full shell model ("shell membranes"), and those which model deformable sheets which *cannot* support moments and transverse shears ("ideal membranes"). For a discussion on the differences between the two types of membranes, see Libai and Simmonds (1988) and Libai (1992).

Many very thin and very flexible shells may be regarded as ideal membranes. These include balloons, thin films, inflatable structures, biological membranes and textiles. In all of these, nonlinear membrane effects dominate the behavior of the shell, so that the moments and transverse shear forces are often considered negligible everywhere.

In this paper we consider a circular membrane tube, which is initially subjected to combined uniform internal pressure and longitudinal extension. These result in a known homogeneous state of stresses accompanied by large homogeneous deformation. Then some changes in the conditions of the problem are considered. These changes may be in the form of additional nonuniform applied loads, or of a local change in the geometry, e.g. a small slit (crack) is introduced, the shape of the boundary is slightly changed or a small hole is punctured in the membrane. We seek the new inhomogeneous deformation and state of stress in the tube.

The membrane material is taken to be hyperelastic. We do not specify the material in detail, but require that if the tube is subjected to a large homogeneous deformation field, then its deformed configuration would remain a circular tube, and that its incremental elasticity from this state would be effectively orthotropic, with lines of symmetry coinciding with the directions of the generators and circles. Materials which are isotropic in the undeformed state fulfil this requirement (Green and Zerna, 1968, Chapter 4), as well as orthotropic materials.

The general equations of nonlinear membrane theory are fairly complicated (Gould, 1987; Green and Adkins, 1970; Leonard, 1988; Libai and Simmonds, 1983; Steigmann, 1990; Libai, 1992). In order to simplify them and make them more amenable for computation, we derive here approximate equations, by regarding the unknown state of stresses in the membrane as a *small* perturbation in the neighborhood of the known homogeneous state of stress produced by the initial pressure and extension. [A similar procedure has been used by Libai (1972) in a different context.] We also derive simplified forms of these equations. In Part II of this paper (Givoli and Libai, 1994), we propose a numerical method to solve the simplified equations, and we present the solution of some specific problems.

The outline of the paper is as follows. We start from the general equations of nonlinear membrane theory in Section 2. Then, the useful approximate equations of small stretchings in the neighborhood of a known state are introduced in Section 3. In this approximation, the nonlinear terms in the incremental strains are neglected, but all of the nonlinear curvature terms (both normal and geodesic) are retained. In Section 4, the equations are specialized to the cylindrical membrane, with the homogeneously deformed cylinder serving as the known reference configuration. The set of equations thus obtained is still highly nonlinear.

To obtain a more manageable first approximation to the state of stress in the perturbed tube, *linearization* about the homogeneous state is performed in Section 5. The linearization process will be performed in stages. It is our intention to show in the process some *partly linearized* theories which find important uses in membrane analysis. Of these, the more important one is the incremental "small-strain-finite-rotation" theory, where linearization is performed on the incremental strains and incremental constitutive relations, but not on the curvatures. It should be emphasized that the curvatures are not constitutively related to the stress resultants or strains, and therefore the assumption of small incremental curvatures is an *additional* assumption beyond that of small incremental strains. Thus, one might have a case of virtually zero strains *and* large changes of curvature, as is evidenced by the case of (almost) inextensional deformations (recall the deformation of a sheet of paper!).

Finally, we arrive at the complete linearization. This is a rather crude approximation, but it still brings forth the essential features of the problem. In Section 6, we introduce two scalar potential functions: the stress function and the curvature function. Then we recast the linear set of equations obtained previously as a set of two partial differential equations expressed in terms of these two potential functions, or alternatively as a single sixth-order equation in terms of the stress function.

The latter equations are reasonably convenient for analysis and further simplifications are not essential. However, in Section 7 we consider for illustrative purposes some simplifying assumptions, namely the assumption of small strains at the homogeneous state and the assumptions associated with Donnell's theory. These assumptions have the effect of preserving the general form of the equations while "symmetrizing" them and simplifying their coefficients. As a special case, the equations for the isotropic material are also given. In Section 8 we discuss the deformation and the appropriate boundary conditions for the general equations and for the simplified equations.

We remark that in Libai (1972), the author has already considered a perturbation and linearization process for membrane shells, and applied it to *noncircular* cylinders. However, his procedure was confined to linear isotropic materials and a scalar curvature potential was *not* introduced. This limited the applicability of his method.

In Section 9 we introduce two variational formulations. The first is based on the weak form of the equations derived in Section 6. This weak form is nonsymmetric, but becomes symmetric when the Donnell approximation (considered in Section 7) is also used. In the latter case, a variational principle is presented as well. The second variational formulation is based on the principle of total potential energy. It is applicable only to the simplified problem based on Donnell's approximation, and it is slightly more complicated than the first variational formulation. However, it enables the use of more general boundary conditions, with clear physical interpretation. Both variational formulations may serve as the starting point for a Galerkin approximation method. We close with some concluding remarks in Section 10.

In Part II of this paper, a numerical method based on the finite element approach is devised for the solution of incremental tube problems. Several examples involving non-uniform loads and boundary disturbances are worked out, and a more general treatment of the attenuation of boundary disturbances is given.

2. NONLINEAR MEMBRANE THEORY

In this paper we use the intrinsic approach to shell theory. In this approach, the use of the position vector, displacement and displacement gradients is *avoided*, and the intrinsic properties of the deformed membrane surface—metric $a_{\alpha\beta}$ and curvature $b_{\alpha\beta}$—are used directly as deformation variables. To these are added the Cauchy (or Kirchhoff) stress resultants $n^{\alpha\beta}$ (or $N^{\alpha\beta}$). Some advantages of the intrinsic method are considered to be its simpler formulation in highly nonlinear problems and the ease of introducing specialized theories. It is widely used in theoretical shell analysis but less so in numerical analysis.

The other method of analysis, which is very widely used in three-dimensional solid mechanics and also finds extensive use in numerical shell analysis, is the displacement approach. Here, the position, displacements and displacement gradients are used to formulate the problem (Hughes and Pister, 1978; Ibrahimbegovic and Gruttmann, 1993). We shell not dwell on the merits and demerits of the approaches, but, as a rule, the displacement approach is more suitable to the numerical analysis of general shell systems, whereas the intrinsic approach appears to be more efficient in some specific problems where approximations related to the problem at hand may lead to great simplification.

The basic field equations of the intrinsic approach have the tensorial form (Gould, 1987; Green and Adkins, 1970; Leonard, 1988; Libai and Simmonds, 1983):

$$n^{\alpha\beta}\|_{\alpha}+p^{\beta}=0 \quad \text{(tangential equilibrium)} \tag{1}$$

$$n^{\alpha\beta}b_{\alpha\beta}+p_{\mathrm{n}}=0 \quad \text{(normal equilibrium)} \tag{2}$$

$$\varepsilon^{\beta\gamma}b_{\alpha\beta}\|_{\gamma}=0 \quad \text{(compatibility; Codazzi)} \tag{3}$$

$$\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(a_{\alpha\beta,\gamma\delta}+a_{\lambda\theta}\Gamma^{\lambda}_{\alpha\beta}\Gamma^{\theta}_{\gamma\delta}+b_{\alpha\beta}b_{\gamma\delta})=0 \quad \text{(compatibility; Gauss)} \tag{4}$$

All quantities and tensorial operators in the equations are referred to the *deformed* configuration. The summation convention with respect to repeated indices is enforced. A vertical double-bar denotes covariant differentiation and a comma denotes partial differentiation. In eqns (1)–(4), $\varepsilon^{\alpha\beta}$ is the surface permutation tensor, the $\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols and $(p^{\alpha}, p_{\mathrm{n}})$ are the components of the loading vector.

To complete the equations, constitutive relations must be appended, which relate the stress resultants $n^{\alpha\beta}$ to the metric components $a_{\alpha\beta}$. Here we consider the relations corresponding to a hyperelastic material. We let $E_{\alpha\beta}$ be the Green strain tensor,

$$E_{\alpha\beta}=\tfrac{1}{2}(a_{\alpha\beta}-a^{*}_{\alpha\beta}), \tag{5}$$

where the $a^{*}_{\alpha\beta}$ are the metric components in the initial (undeformed) configuration. Also, we let $\tilde{N}^{\alpha\beta}$ be the Kirchhoff stress resultant component measured per unit undeformed length, but in deformed directions,

$$\tilde{N}^{\alpha\beta}=\left(\frac{a}{a^{*}}\right)^{1/2}n^{\alpha\beta}. \tag{6}$$

Here, $a=\det(a_{\alpha\beta})$ and $a^{*}=\det(a^{*}_{\alpha\beta})$. Finally, we let $W(E_{\alpha\beta})$ be the strain energy density function, per unit undeformed area. Then the hyperelastic constitutive equations have the form:

$$\tilde{N}^{\alpha\beta} = \frac{\partial W}{\partial E_{\alpha\beta}}. \tag{7}$$

In this paper, we assume that wrinkle fields are not formed in the membrane. This is accomplished by restricting the shell thickness h, if necessary, to $h < h_{cr}$, where h_{cr} is the critical thickness of local compressive (or shear) buckling for the case at hand. For the general problem of wrinkle fields, see Steigmann (1990).

Equations (1)–(4) and (7) are a set of nine equations in the nine field variables $n^{\alpha\beta}, b_{\alpha\beta}$ and $a_{\alpha\beta}$. Together with appropriate boundary conditions, they constitute a mathematical problem in general terms. The data needed for the nonlinear membrane problem are the surface loading (in the deformed directions), conditions on the deformed boundary (shape or tractions), specification of the material (in the form of a strain energy density function W) and the *metric* of the undeformed membrane. It should be emphasized that, except for its metric, the undeformed configuration *is not needed* at all in the analysis of an ideal membrane. (As an example, a sheet of paper can be bent prior to loading into a tube of whatever cross-section, or even squeezed into a square package, and still have the same deformed configuration after the loading is applied.)

3. INCREMENTAL APPROACH

In most cases the shell equations are expressed either with respect to the undeformed configuration, or with respect to the current (deformed) configuration, as in eqns (1)–(4). However, sometimes it is useful to express the equations with respect to a certain *known* deformed state, which will be termed *the reference state*. Thus, there are three distinct states: the undeformed (initial) state, the reference state and the deformed (final) state. Referring the equations to the reference state is not associated by itself with any approximation; it merely leads to a new set of equations which is equivalent to the original one, only expressed in a different manner. Of course, it is tacitly assumed that the three states can be joined by suitable equilibrium paths in the solution domain.

In some cases it is justified to regard the deformed state and the reference state as *neighboring states*, in that the increments *in the metric* between the two states are small. This makes it possible to suppress the nonlinear terms in the incremental strains. The procedure is explained in Libai and Simmonds (1983) and Libai (1972). Here we apply it to the membrane equations (1)–(4). Note that while we drop the nonlinear stretching terms in the incremental metric, we retain all the nonlinear terms associated with changes in the normal and geodesic curvatures, multiplied by the stress resultants as well as the nonlinear terms in the curvatures. Thus, although the procedure leads to substantial simplification compared with the most general system, the equations still remain highly nonlinear.

In order to write eqns (1)–(4) in terms of the known reference state, we first define $e_{\alpha\beta}$ to be the incremental strain component and $N^{\alpha\beta}$ to be the (contravariant) components of the stress resultant tensor at the current configuration, but per unit length of the reference state, i.e.

$$e_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\beta} - \bar{a}_{\alpha\beta}), \quad N^{\alpha\beta} = \left(\frac{a^*}{\bar{a}}\right)^{1/2} \tilde{N}^{\alpha\beta}. \tag{8}$$

Here the $\bar{a}_{\alpha\beta}$ are the metric components in the reference state and $\bar{a} = \det(\bar{a}_{\alpha\beta})$. Then eqns (1)–(4) are reduced to the following form (Libai and Simmonds, 1983; Libai, 1972):

$$N^{\alpha\beta}|_\alpha + N^{\alpha\theta}(2e^\beta_\alpha|_\theta - e_{\alpha\theta}|^\beta) + \bar{p}^\beta = 0 \quad \text{(tangential equilibrium)} \tag{9a}$$

$$N^{\alpha\beta} b_{\alpha\beta} + \bar{p}_\text{n} = 0 \quad \text{(normal equilibrium)} \tag{9b}$$

$$\varepsilon^{\beta\gamma}[b_{\alpha\beta}|_\gamma - b_{\theta\beta}(e^\theta_\alpha|_\gamma + e^\theta_\gamma|_\alpha - e_{\alpha\gamma}|^\theta)] = 0 \quad \text{(compatibility; Codazzi)} \tag{10a}$$

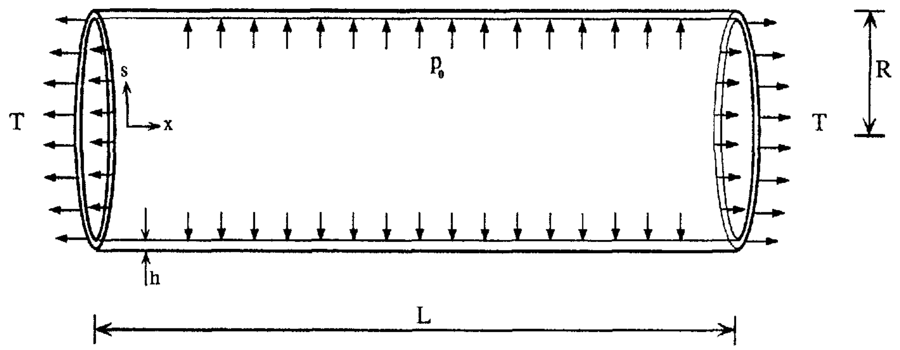

Fig. 1. Geometry of the membrane tube and applied loads in the reference (homogeneously deformed) configuration.

$$\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(e_{\alpha\beta}|_{\gamma\delta}+\tfrac{1}{2}b_{\alpha\beta}b_{\gamma\delta}) = \bar{K}(1-e^{\alpha}_{\alpha}) \quad \text{(compatibility; Gauss).} \tag{10b}$$

Here, a single vertical bar indicates covariant differentiation with respect to the metric $\bar{a}_{\alpha\beta}$ of the reference state. Also, the loads $\bar{p}^{\beta}$ and $\bar{p}_n$ are measured per unit area of the reference state, and $\bar{K}$ is its Gaussian curvature.

Equations (9) and (10) are accompanied by hyperelastic constitutive relations analogous to eqn (7), which relate $e_{\alpha\beta}$ to $N^{\alpha\beta}$. These can also be linearized with respect to the incremental strains by using a formal expansion of the the derivatives of the strain energy density W about the reference state. We use the subscripts (2) and (1) to denote quantities evaluated at the final and reference states, respectively. Then,

$$\left(\frac{\partial W}{\partial E_{\alpha\beta}}\right)_{(2)} = \left(\frac{\partial W}{\partial E_{\alpha\beta}}\right)_{(1)} + \left(\frac{\partial^2 W}{\partial E_{\alpha\beta}\partial E_{\gamma\delta}}\right)_{(1)} e_{\gamma\delta} + \mathscr{R}, \tag{11}$$

where in the linearized case we suppress the remainder $\mathscr{R}$. Now we multiply through by $(a^*/\bar{a})^{1/2}$, so that all quantities are measured *per unit length of the reference surface*. Then we get from eqn (11) the linearized constitutive equation,

$$N^{\alpha\beta} = N^{\alpha\beta}_{(1)} + s^{\alpha\beta\gamma\delta}e_{\gamma\delta}. \tag{12}$$

Here $N^{\alpha\beta}_{(1)} = (a^*/\bar{a})^{1/2}(\partial W/\partial E_{\alpha\beta})_{(1)}$ are the (known) Kirchhoff components of the stress resultant tensor at the reference configuration, and $s^{\alpha\beta\gamma\delta} = (a^*/\bar{a})^{1/2}(\partial^2 W/\partial E_{\alpha\beta}\partial E_{\gamma\delta})_{(1)}$ are the elastic coefficients at the reference state. Note that these coefficients are functions of the *known* strains of the reference state. The symmetry property $s^{\alpha\beta\gamma\delta} = s^{\gamma\delta\alpha\beta}$ follows from the definition, but the linearized constitutive equations (12) need not be isotropic even if the undeformed material is.

Equation (12) can be substituted back into eqns (9) and the results linearized in the incremental strains. To avoid excessive rewriting, we defer the substitution to a later stage. A special variant of the equations is obtained by suppressing the strains in eqns (9) and taking the $N^{\alpha\beta}$ to be "reactive resultants." This is the so-called "membrane inextensional" behavior which has important applications. We shall not develop this case here.

4. THE MEMBRANE TUBE PROBLEM—INCREMENTAL FORMULATION

Now we state the specific problem under consideration. We are given a circular membrane tube with the initial dimensions of length L_0, thickness h_0 and radius R_0, subjected to a uniform internal pressure p_0 (per unit deformed area) and a pulling force T, such that its dimensions after loading are L, h and R (see Fig. 1). The inequality $h/r \ll 1$ is assumed to hold, so that the membrane may be regarded as very thin.

The material of the membrane is taken to be homogeneous, hyperelastic and orthotropic, with lines of material symmetry coinciding with the generators and circles of the tube. The deformation due to pulling and pressurization may be large, but is homogeneous, so that there is no inherent difficulty in calculating L, h and R. It is assumed that such analysis was done, so that L, h and R are available.

At this stage, we consider some changes in the conditions of the tube. To fix ideas, we look at two types of changes:

(a) Additional pressure loading p is applied on the tube. This load is not uniform, and may be positive or negative (or zero) at different locations on the tube surface. Also, we allow p to be nonsmooth. In fact, later we shall consider applied concentrated point forces (see Part II: Givoli and Libai, 1994), by using p which is the Dirac delta.

(b) Local change in the geometry is introduced. For example, a small slit (crack) is formed, the shape of the boundary is slightly changed or a small hole is punctured in the tube.

These two types of changes may occur separately or in a certain combination. We assume that the total additional load is *self-equilibrated*. These changes lead to a new configuration, associated with an inhomogeneous deformation and a new state of stress. We seek this new configuration, within the framework of nonlinear membrane theory.

Now we specialize eqns (9) and (10) to the tube problem under consideration. The homogeneously deformed state of the tube, under combined tension and pressurization, is chosen to be the reference state. The final state is produced by the additional loads and/or change in geometry. The loading in the present case is $\bar{p}^{\alpha} = 0$, $\bar{p}_{\text{n}} = -(p_0+p)$ per unit reference area. In the case of a slit, it is realized that large strains may develop close to the corners of the slit, but the final state may still be regarded as an approximation in the sense of linear elastic fracture mechanics.

The material coordinate system on the cylindrical surface is denoted by (x, s), where x and s are distances at the reference state along the generators and parallel circles, respectively (see Fig. 1). With this choice, the metric of the reference state is Cartesian, and $\bar{a}_{\alpha\beta} = \delta_{\alpha\beta}$ (Kronecker's delta). Also, $\bar{K} = 0$, and partial derivatives replace covariant derivatives. With this in mind, eqns (9) and (10) become

$$N^{xx}_{,x} + N^{ss}_{,s} + N^{xx}e_{xx,x} + 2N^{xs}e_{xx,s} + N^{ss}(2e_{xs,s} - e_{ss,x}) = 0 \tag{13}$$

$$N^{xs}_{,x} + N^{ss}_{,s} + N^{xx}(2e_{xs,x} - e_{xx,s}) + 2N^{xs}e_{ss,x} + N^{ss}e_{ss,s} = 0 \tag{14}$$

$$N^{xx}b_{xx} + 2N^{xs}b_{xs} + N^{ss}b_{ss} = p_0 + p \tag{15}$$

$$b_{xx,s} - b_{xs,x} - b_{xx}e_{xx,s} - b_{xs}(e_{ss} - e_{xx})_{,s} + b_{ss}(2e_{xs,x} - e_{xx,s}) = 0 \tag{16}$$

$$b_{ss,x} - b_{xs,s} - b_{ss}e_{ss,x} - b_{xs}(e_{xx} - e_{ss})_{,s} + b_{xx}(2e_{xs,s} - e_{ss,x}) = 0 \tag{17}$$

$$b_{xx}b_{ss} - b_{xs}b_{xs} + e_{xx,ss} + e_{ss,xx} - 2e_{xs,xs} = 0. \tag{18}$$

Equations (13)–(18) are accompanied by three constitutive equations relating e_{xx}, e_{xs}, e_{ss} to N^{xx}, N^{xs}, N^{ss}, as discussed before. This constitutes a system of nine equations with nine unknown variables.

5. LINEARIZATION

Even in the simplified form of eqns (13)–(18), the problem is very complicated both analytically and numerically, due to the presence of highly nonlinear effects. Further simplification can be achieved by assuming that the increments from the reference state are sufficiently small so as to permit complete linearization in these increments. This approximation, although far from being exact, preserves the main features of membrane behavior and facilitates the application of analytical and numerical techniques to the reduced problem. The approach of "small deformations superposed on large" [see Green and Zerna (1968, Chapter 4)] has many applications in nonlinear mechanics, and has been used to investigate stability and vibration of membranes in Budiansky (1968), Koga (1972), Shield (1972) and Needleman (1976).

To perform the linearization, we first define the *incremental* stress resultants and curvatures as the differences between the values of these variables at the final state and the corresponding values at the reference state:

$$N_x = N^{xx} - P_x, \quad N_s = N^{ss} - P_s, \quad N_t = N^{xs} - 0 \tag{19}$$

$$\kappa_x = b_{xx} - 0, \quad \kappa_s = b_{ss} - \frac{1}{R}, \quad \kappa_t = b_{xs} - 0. \tag{20}$$

Here,

$$P_x = \frac{T}{2\pi R}, \quad P_s = p_0 R \tag{21}$$

are the known stress resultants due to the applied tension and pressure loads, respectively, per unit length of the reference state. [Note that the Ps in eqns (21) are in fact the quantities denoted $N^{\alpha\beta}_{(1)}$ in eqn (12).] The incremental strains were already defined in eqn (8), but we also introduce the notation,

$$\varepsilon_x \equiv e_{xx}, \quad \varepsilon_s \equiv e_{ss}, \quad \gamma_{xs} \equiv 2e_{xs}. \tag{22}$$

Now we substitute eqns (19)–(22) into eqns (13)–(18) and drop nonlinear terms. This yields

$$(N_x - P_s\varepsilon_s + P_x\varepsilon_x)_{,x} + (N_t + P_s\gamma_{xs})_{,s} = 0 \tag{23}$$

$$(N_s - P_x\varepsilon_x + P_s\varepsilon_s)_{,s} + (N_t + P_x\gamma_{xs})_{,x} = 0 \tag{24}$$

$$P_x\kappa_x + P_s\kappa_s + \frac{1}{R}N_s + \underline{N_x\kappa_x} + \underline{N_s\kappa_s} + \underline{2N_t\kappa_t} = p \tag{25}$$

$$\left(\kappa_x - \frac{1}{R}\varepsilon_x\right)_{,s} - \left(\kappa_t - \frac{1}{R}\gamma_{xs}\right)_{,x} = 0 \tag{26}$$

$$\left(\kappa_s - \frac{1}{R}\varepsilon_s\right)_{,x} - \kappa_{t,s} = 0 \tag{27}$$

$$\frac{1}{R}\kappa_x + \varepsilon_{x,ss} + \varepsilon_{s,xx} - \gamma_{xs,xs} + \underline{\kappa_x\kappa_s} - \underline{\kappa_t^2} = 0. \tag{28}$$

In eqns (25) and (28) we show some underlined nonlinear terms which form the basis for a *small-strain-finite-rotation incremental theory*. We shall refer to these terms in the sequel. However, for a complete linearization we shall drop these terms from now on.

We also linearize the constitutive equations (7), namely we consider linear relations between the *incremental* stresses and strains. Again, this is justified if the reference state and the final state are sufficiently close to each other. In the present case, due to the orthotropic properties of the tube material and the homogeneous nature of the reference state, we have the relations

$$\varepsilon_x = c_x N_x - c_{xs} N_s \tag{29}$$

$$\varepsilon_s = c_s N_s - c_{sx} N_x \tag{30}$$

$$\gamma_{xs} = c_t N_t. \tag{31}$$

Here c_x, c_s, c_{xs}, c_{sx} and c_t are given elastic coefficients ("compliances") which are functions of the strains of the reference state (the latter may be large). For homogeneous materials and homogeneous deformations of the reference state, these coefficients do not vary with position. The matrix of these c coefficients is in fact the inverse of the matrix of the s coefficients in eqn (12) and is, therefore, symmetric. This symmetry property is not preserved for many other measures of incremental stress. An example for Cauchy stresses is given by Green and Zerna (1968, Chapter 4). A more general discussion is given by Simo and Pister (1984). A discussion of materials having the property of coincident directions of stress and strain is given by Ibrahimbegovic and Gruttmann (1993).

Equations (23)–(31) are nine *linear* equations in the nine incremental variables defined in eqns (19), (20) and (22). For P_x and P_s of the same sign (which is assumed here), the linearized form constitutes a sixth-order elliptic system. It is important to emphasize that although this system is linear, it does reflect the "nonlinear behavior" of the membrane through the appearance of the initial loads P_x and P_s in the coefficients of eqns (23)–(25). This became possible only due to the introduction of the reference state in the previous section. In contrast, the direct full linearization of the membrane equations (1)–(4) and (7) *around the undeformed state* would fail to capture any nonlinear effects.

We also remark that in the axisymmetric case, eqns (23)–(31) are independent of s and reduce to a single second-order differential equation in x (plus two quadratures). However, we shall be concerned with the more general situation.

6. STRESS AND CURVATURE FUNCTION FORMULATION

Now we recast eqns (23)–(31) as two equations in two unknown scalar functions, or as a single sixth-order equation. To this end, we introduce the *stress function* ϕ and the *curvature function* ψ, defined by the relations

$$N_t = -\phi_{,xs} \quad \kappa_t = \psi_{,xs}. \tag{32}$$

Other choices are possible too, but they lead to equivalent results. We use eqns (29)–(31), (23), (24) and (32), and after some algebra we obtain the following expressions for the incremental stress resultants and strains:

$$N_x = \alpha_x \phi_{,ss} + \alpha_{xs} \phi_{,xx} \tag{33}$$

$$N_s = \alpha_s \phi_{,xx} + \alpha_{sx} \phi_{,ss} \tag{34}$$

$$\varepsilon_x = \beta_x \phi_{,ss} - \beta_{xs} \phi_{,xx} \tag{35}$$

$$\varepsilon_s = \beta_s \phi_{,xx} - \beta_{sx} \phi_{,ss} \tag{36}$$

$$\gamma_{xs} = -c_t \phi_{,xs}. \tag{37}$$

Based on these results, we get expressions for the curvatures from eqns (26) and (27):

$$\kappa_s = \psi_{,ss} + \frac{1}{R}(\beta_s \phi_{,xx} - \beta_{sx}\phi_{,ss}) \tag{38}$$

$$\kappa_x = \psi_{,xx} + \frac{1}{R}[\beta_x \phi_{,ss} + (c_t - \beta_{xs})\phi_{,xx}]. \tag{39}$$

The coefficients in eqns (33)–(39) are defined as follows:

$$\alpha_x = \frac{1}{D}(1+P_s c_t)(1+a_s), \quad \alpha_s = \frac{1}{D}(1+P_x c_t)(1+a_x) \tag{40}$$

$$\alpha_{xs} = \frac{1}{D}(1+P_x c_t)a_s, \quad \alpha_{sx} = \frac{1}{D}(1+P_s c_t)a_x \tag{41}$$

$$a_x = P_x c_x + P_s c_{sx}, \quad a_s = P_s c_s + P_x c_{xs}, \quad D = 1 + a_x + a_s \tag{42}$$

$$\beta_x = \alpha_x c_x - \alpha_{sx} c_{xs}, \quad \beta_{xs} = \alpha_x c_{xs} - \alpha_{xs} c_x \tag{43}$$

$$\beta_{sx} = \alpha_s c_{sx} - \alpha_{sx} c_s, \quad \beta_s = \alpha_s c_s - \alpha_{xs} c_{sx}. \tag{44}$$

With these definitions, it is easy to verify that eqns (23), (24), (26), (27) and (29)–(31) are all satisfied *identically*. This leaves out eqns (25) and (28). The former, which is an equilibrium equation, yields

$$P_x \psi_{,xx} + P_s \psi_{,ss} + q_x \phi_{,xx} + q_s \phi_{,ss} = p, \tag{45}$$

where

$$q_x \equiv \frac{1}{R}[(c_t - \beta_{xs})P_x + \beta_s P_s + \alpha_s], \quad q_s \equiv \frac{1}{R}(\beta_x P_x - \beta_{sx} P_s + \alpha_{sx}). \tag{46}$$

Equation (28), which is a compatibility equation, gives

$$R\nabla_*^4 \phi + \frac{1}{R}(c_t - \beta_{xs})\phi_{,xx} + \frac{1}{R}\beta_x \phi_{,ss} + \psi_{,xx} = 0. \tag{47}$$

Here, we use the "modified" biharmonic operator defined by

$$\nabla_*^4 \phi \equiv \beta_x \phi_{,ssss} + \beta_s \phi_{,xxxx} + \rho \phi_{,xxss}, \tag{48}$$

where the constant ρ is given by

$$\rho = c_t - \beta_{xs} - \beta_{sx}. \tag{49}$$

Thus, the problem is stated by the two equations (45) and (47), in terms of the stress function ϕ and the curvature function ψ.

It is also possible to derive a single equation in ϕ. This is done by applying each of the operators $P_x \partial^2/\partial x^2$ and $P_s \partial^2/\partial s^2$ separately to eqn (47), summing the two results and using eqn (45) to eliminate $P_x \psi_{,xxxx} + P_s \psi_{,xxss}$. The final result is

$$R^2 \nabla_*^4 (P_x \phi_{,xx} + P_s \phi_{,ss}) - (\alpha_s + P_s \beta_s)\phi_{,xxxx} + P_s \beta_x \phi_{,ssss} + [P_s(\beta_{sx} - \beta_{xs} + c_t) - \alpha_{sx}]\phi_{,xxss} = -Rp_{,xx}. \tag{50}$$

This is a sixth-order linear differential equation (with constant coefficients) in the stress function. The coefficients are, of course, functions of the strains of the reference state. Note that the second axial derivative of the pressure p appears on the right hand side of eqn (50), whereas in eqn (45) only p itself appears. Thus, if p is not smooth in the x direction, the two-equation formulation (45) and (47) may be advantageous.

In the substitutions leading to eqns (45) and (47), we deleted the nonlinear (underlined) terms in eqns (25) and (28). Their retention would have resulted in a ϕ–ψ version of the finite rotation incremental membrane theory, with additional nonlinear terms in eqns (45) and (47).

7. FURTHER SIMPLIFICATIONS

Equations (45) and (47), or eqn (50), are the final equations proposed in this paper, and should be used as the starting point for analysis. Further simplifications are not needed. However, in this section we consider the equations that eqns (45), (47) and (50) reduce to if one chooses to make some additional simplifying assumptions. These assumptions have the effect of preserving the general form of the equations while simplifying their coefficients. We emphasize that the simplifications considered here are illustrative but do not pose restrictions on the theory developed above.

The most obvious assumption is that of *small strains at the reference* (*homogeneous*) *state*, which applies in many cases. Mathematically, the products $P_i c_j$ are assumed to be much smaller than one, for all of P_x, P_s, c_x, c_s, c_{xs}, c_{sx} and c_t. These products are neglected in the equations with respect to much larger terms, although they still remain in some of the terms where they are the dominating factor. Thus, the inherent geometric nonlinearity of the homogeneous state is retained. This situation in somewhat similar to that of simple beam buckling, which is governed by the linear equation $EIw^{(iv)}+Pw''=0$; even for small strains there remains a "nonlinear effect" which is manifested by the term Pw''. In the present context, the coefficients (40)–(44) become

$$\alpha_x = \alpha_s = 1\,, \quad \alpha_{xs} = a_s\,, \quad \alpha_{sx} = a_x \tag{51}$$

$$a_x = P_x c_x + P_s c_{sx}\,, \quad a_s = P_s c_s + P_x c_{xs} \tag{52}$$

$$\beta_x = c_x\,, \quad \beta_{xs} = c_{xs}\,, \quad \beta_{sx} = c_{sx}\,, \quad \beta_s = c_s. \tag{53}$$

Then eqn (50) reduces to

$$R^2\nabla_*^4(P_x\phi_{,xx}+P_s\phi_{,ss})-\phi_{,xxxx}+P_s c_x\phi_{,ssss}+(P_s c_t-P_x c_x-P_s c_{sx})\phi_{,xxss} = -Rp_{,xx}. \tag{54}$$

A more extensive simplification is achieved by making the assumptions associated with *Donnell's theory* of shells (Donnell, 1976; Libai and Simmonds, 1988; Niordson, 1985). In the present context the assumptions are

$$|N_x|\,, \quad |N_s| \gg \max\,(P_x|\varepsilon_x|\,;\, P_s|\varepsilon_s|)\,, \quad |N_t| \gg \max\,(P_x|\gamma_{xs}|\,;\, P_s|\gamma_{xs}|) \tag{55}$$

$$|\kappa_x| \gg (1/R)|\varepsilon_x|\,, \quad |\kappa_s| \gg (1/R)|\varepsilon_s|\,, \quad |\kappa_t| \gg (1/R)|\gamma_{xs}|. \tag{56}$$

Thus, all terms but the first one in each of the parentheses of eqns (23), (24), (26) and (27) are suppressed. This simplification is known to be justified in many (but not all) cases and to preserve the essential features of shell behavior (Libai and Simmonds, 1988; Niordson, 1985). It leads to the following simple expressions for the stresses and curvatures which replace (32)–(34), (38) and (39):

$$N_x = \phi_{,ss}, \quad N_s = \phi_{,xx}, \quad N_t = -\phi_{,xs} \tag{57}$$

$$\kappa_x = \psi_{,xx}, \quad \kappa_s = \psi_{,ss}, \quad \kappa_t = \psi_{,xs}. \tag{58}$$

Using eqns (57) and (58), the two linear equations (45) and (47) reduce to

$$P_x\psi_{,xx} + P_s\psi_{,ss} + \frac{1}{R}\phi_{,xx} = p \tag{59}$$

$$R\nabla_*^4\phi + \psi_{,xx} = 0. \tag{60}$$

Also, eqn (50) reduces to

$$R^2\nabla_*^4(P_x\phi_{,xx} + P_s\phi_{,ss}) - \phi_{,xxxx} = -Rp_{,xx}. \tag{61}$$

The results above hold for an orthotropic material whose constitutive behavior is given by eqns (29)–(31). Now we specialize these results to the case of an *isotropic* material. In this case, the elastic compliances in eqns (29)–(31) are given by

$$c_x = c_s = \frac{1}{Eh}, \quad c_{xs} = c_{sx} = \frac{\nu}{Eh}, \quad c_t = \frac{2(1+\nu)}{Eh}, \tag{62}$$

where E is Young's modulus and ν is Poisson's ratio. Then, it is easy to verify that eqns (59)–(61) (for the orthotropic case) remain the same, except that the "modified" biharmonic operator ∇_*^4 is reduced to $(1/Eh)\nabla^4$, where ∇^4 is the proper biharmonic operator.

8. DEFORMATION AND BOUNDARY CONDITIONS

After solving the mathematical problem stated above and finding the stresses in the membrane tube, it is sometimes also desirable to obtain the displacements. This is quite difficult to achieve in the most general case. However, if the final state is "close" to the reference state in that the incremental rotations are also small, then the incremental tangential displacement in the s direction, v_s, and the incremental normal displacement, w, are related to the geodesic curvature κ_{gx} and to the normal curvature κ_x via

$$\frac{\partial^2 v_s}{\partial x^2} = \kappa_{gx}; \quad \frac{\partial^2 w}{\partial x^2} = \kappa_x, \tag{63}$$

where, in the present case,

$$\kappa_{gx} = \gamma_{xs,x} - \varepsilon_{x,s}. \tag{64}$$

Thus, the tangential and normal displacements may be obtained by twice integrating with respect to x the geodesic and normal curvatures, respectively.

Now, *if we adopt the Donnell approximation* introduced in the previous section, then we may use eqns (58) and (63) to conclude that the curvature function $\psi(x, s)$ has in fact the physical interpretation of the *normal displacement*, i.e. $\psi = w$. Note that even if the Donnell approximation is not valid along the entire tube, it may be justified at and near an edge which is fixed in the normal direction. Thus, one may represent such an edge by the boundary condition $\psi = 0$. In more general cases, when small rotations and displacements cannot be assumed, the boundary conditions should be expressed in terms of strains and curvatures. This can be done along the lines shown by Libai and Bert (1994).

Now we consider appropriate boundary conditions for the tube problem. We discuss four different types of boundary conditions: those on each of the two edges, symmetry conditions, the conditions on the boundary of a circumferential slit and those on the boundary of a longitudinal slit. We consider these boundary conditions for the general problems (Sections 2–4), as well as for the linearized problems (Sections 5–7).

8.1. *Boundary conditions at the two edges*

Suppose that on the two edges all the field variables assume their known reference state values. Under the general theory [Sections 2–4, eqns (1)–(18)], the appropriate boundary conditions are $n_x^x = P_x, n_x^s = 0$. Note that only *two* conditions are needed. This is due to the "parabolicity" of the stress operator along the edge [see Libai and Simmonds (1983)]. On the other hand, each of the simplified problems [Sections 5–7, eqns (19)–(62)] constitutes a sixth-order linear elliptic partial differential system, and thus requires *three* boundary conditions on each edge. The conditions $\varepsilon_s = 0$, $\kappa_s = 0$ and $N_x = 0$ are acceptable, although other conditions are possible too.

If the (ϕ, ψ) formulation (59) and (60) is used, three appropriate conditions are

$$\phi = 0; \quad \phi_{,xx} = 0; \quad \psi = 0. \tag{65}$$

These conditions are those of a "diaphragm" (simple) support, which requires that no in-plane distortions take place, and that the normal stress does not change. The actual conditions are $\varepsilon_s = \kappa_s = N_x = 0$, but the use of eqns (33), (36) and (38) leads, after twice integrating with respect to s, to eqns (65). These conditions also have some numerical advantage [see Section 9 and Givoli and Libai (1994)].

8.2. *Symmetry boundary conditions*

Suppose that the tube may be divided into two halves in the longitudinal direction, such that the plane $x = L/2$ is a plane of symmetry of the problem. Then, to reduce the computational effort in a numerical analysis, only the half $0 \leqslant x \leqslant L/2$ is to be considered. On the edge $x = L/2$, symmetry boundary conditions must be imposed. Acceptable symmetry conditions in the general case are $n_x^s = 0$ and either of $n_{x,x}^x = 0$ or $n_{s,x}^s = 0$. In the linearized incremental problem, the three conditions $N_t = 0$, $N_{x,x} = 0$ and $N_{s,x} = 0$ may be used. In the (ϕ, ψ) formulation (59) and (60), appropriate symmetry conditions are $\phi_{,x} = 0$, $\phi_{,xxx} = 0$ and $\psi_{,x} = 0$ (see Section 9).

Similarly, suppose that the tube may be divided into two half-cylinders, so that the plane that passes through the longitudinal lines $s = 0$ and $s = \pi R$ is a plane of symmetry of the problem. Again, it is beneficial to consider only half of the tube in the analysis, while imposing symmetry boundary conditions on the two new boundaries. This case is analogous to the previous one, with x and s interchanged.

8.3. *Conditions on the boundary of a circumferential slit*

Suppose that one of the "geometrical changes" considered in Section 4 is in the form of a small circumferential slit (crack), which is introduced on the surface of the tube. Here we discuss the appropriate boundary conditions to be applied on the free boundary of such a slit. Under the general theory, the two necessary requirements are $n_x^x = n_x^s = 0$. A consequence of these conditions [see eqn (2) or (15)] is that the equation

$$n_s^s b_s^s = p_0 + p \tag{66}$$

is satisfied along the edge. This equation does not present a new independent condition.

In the linearized incremental case, the two requirements stated above become $N_x = -P_x$ and $N_t = 0$. However, eqn (66) leads to a third condition which is *not* satisfied automatically. From eqns (66) and (19)–(21), and after dropping the nonlinear term $N_s\kappa_s$, we get

$$P_s\kappa_s + \frac{1}{R}N_s = p. \tag{67}$$

Thus, we have three boundary conditions on the edge of the slit, which is the appropriate number of conditions for a sixth-order system, as discussed previously. In the (ϕ, ψ) formulation (59) and (60), the three conditions become

$$\phi_{,ss} = -P_x; \quad \phi_{,xs} = 0; \quad P_s\psi_{,ss} + \frac{1}{R}\phi_{,xx} = p. \tag{68}$$

It should be emphasized that near the edges of the slit the incremental stress resultants are of the same order of magnitude as the prestressing itself, thus violating *locally* one of the requirements of small deformations superposed on large. This effect is expected to die out away from the edge. Thus, in the immediate neighborhood of the slit, the linearized equations should be regarded as a first approximation only, and higher-order terms in the perturbation expansion about the reference configuration are needed if a more accurate analysis is desired (Libai, 1972, p. 929). The problem is of lesser importance if the pre-stressing strains are small.

8.4. *Conditions on the boundary of a longitudinal slit*

Now consider a small slit in the longitudinal direction. Formally, this case is similar to the previous one. Under the general theory, the two necessary requirements are $n_s^s = n_s^x = 0$. A consequence of these conditions [see eqn (2) or (15)] is that the equation

$$n_x^x b_x^x = p_0 + p \tag{69}$$

is satisfied along the edge. As in the case of eqn (66), this equation does not present a new independent condition. However, it has an important physical implication; it states that along the boundary of the slit the entire pressure loading has to be balanced by the longitudinal stress resultant n_x^x alone. Neither it nor the curvature b_x^x can vanish along the boundary. It should be noted that $b_x^x = 0$ in both the undeformed and reference (homogeneous) states. Therefore, equilibrium of the slitted shell cannot be maintained in these geometries, and additional deformation, in the form of "opening" and "bulging", must take place before equilibrium is reached. The implication is that the state of stress near a longitudinal slit cannot be solved by standard linear membrane theory even as a first approximation, since the inclusion of nonlinear effects is crucial in this case. Note that this difficulty does not occur in the case of a circumferential slit.

We note that the "opening" deformation around the slit is responsible for the presence of an unbalanced force. Thus, balancing forces must be introduced at other locations of the tube. One way to introduce such balancing forces is by applying appropriate incremental pressure p, possibly in the form of "reaction" concentrated forces. Another possibility is to consider an additional longitudinal slit, diametrically opposite to the first slit, so that the forces exerted by the two slits balance each other.

In the linearized incremental case, the two requirements become $N_s = -P_s$ and $N_t = 0$. Equation (69) leads, after we drop the nonlinear term $N_x\kappa_x$, to the third condition,

$$P_x\kappa_x = p_0 + p. \tag{70}$$

In the (ϕ, ψ) formulation (59) and (60), the three conditions become

$$\phi_{,xx} = -P_s; \quad \phi_{,xs} = 0; \quad P_x\psi_{,xx} = p_0 + p. \tag{71}$$

The remarks made at the end of Section 8.3 are valid for this case too. However, due to the bulging problem, a second perturbation based on the small-strain-finite-rotation version [eqns (23)–(28), with the underlined nonlinear terms retained] would be desirable even if the prestressing is small.

9. VARIATIONAL FORMULATIONS FOR THE TUBE PROBLEM

In this section we present two variational formulations for the membrane tube problem, expressed via ϕ and ψ. Both formulations may serve as the starting point for a Galerkin-type approximation method. To fix ideas, we assume that the tube is circumferentially "complete", namely contains no slits or holes, and that there are no discontinuities around the circumference. Partly open tubes require additional boundary "jump" terms along the generator boundaries which we take to be zero here. We also assume that the "diaphragm" boundary conditions (65) are imposed on the two edges. Later we shall also consider other boundary conditions.

We denote the domain of the tube by D. We also define the two functions spaces $\mathscr{S}_\phi[D]$ and $\mathscr{S}_\psi[D]$ by

$$\mathscr{S}_\phi[D] = \{\phi \mid \phi \in H^2[D], \quad \phi = 0 \quad \text{at} \quad x = 0, L\} \tag{72}$$

$$\mathscr{S}_\psi[D] = \{\psi \mid \psi \in H^1[D], \quad \psi = 0 \quad \text{at} \quad x = 0, L\}. \tag{73}$$

Here H^n is the nth-order Sobolev space, containing all functions which are in L_2 (i.e. square integrable), and whose derivatives up to order n are also in L_2. Note that the smoothness requirement built in the space $\mathscr{S}_\psi$ is weaker than that built in the space $\mathscr{S}_\phi$.

9.1. *First variational formulation*

The first variational formulation is based on the weak form of eqns (45) and (47). For convenience, we define the following forms:

$$A(\bar{\phi}, \phi) = \int_D \left(\bar{\phi}_{,ss}\beta_x\phi_{,ss} + \bar{\phi}_{,xx}\beta_s\phi_{,xx} + \bar{\phi}_{,ss}\frac{\rho}{2}\phi_{,xx} + \bar{\phi}_{,xx}\frac{\rho}{2}\phi_{,ss} - \bar{\phi}_{,x}\frac{1}{R^2}(c_t - \beta_{xs})\phi_{,x} - \bar{\phi}_{,s}\frac{\beta_x}{R^2}\phi_{,s} \right) \mathrm{d}D \tag{74}$$

$$B(\bar{\phi}, \psi) = -\int_D \bar{\phi}_{,x}\frac{1}{R}\psi_{,x}\,\mathrm{d}D \tag{75}$$

$$C(\bar{\psi}, \psi) = -\int_D (\bar{\psi}_{,x}P_x\psi_{,x} + \bar{\psi}_{,s}P_s\psi_{,s})\,\mathrm{d}D \tag{76}$$

$$D(\bar{\psi}, \phi) = -\int_D (\bar{\psi}_{,x}q_x\phi_{,x} + \bar{\psi}_{,s}q_s\phi_{,s})\,\mathrm{d}D \tag{77}$$

$$F(\bar{\psi}) = \int_D \bar{\psi}p\,\mathrm{d}D. \tag{78}$$

The forms (74)–(77) are symmetric and bilinear, and eqn (78) is linear. Now we consider the problem of finding $\phi \in \mathscr{S}_\phi$ and $\psi \in \mathscr{S}_\psi$ which satisfy

$$A(\bar{\phi}, \phi) + B(\bar{\phi}, \psi) = 0 \quad \forall \bar{\phi} \in \mathscr{S}_\phi, \tag{79}$$

$$D(\bar{\psi}, \phi) + C(\bar{\psi}, \psi) = F(\bar{\psi}) \quad \forall \bar{\psi} \in \mathscr{S}_\psi. \tag{80}$$

It can be shown that this problem is the equivalent weak form of the problem consisting of eqns (45), (47) and (65).

Note that the problem (79) and (80) is not symmetric, in that the forms $B(\cdot,\cdot)$ and $D(\cdot,\cdot)$ are different. Therefore, this problem does not necessarily lead to a variational principle. One may say that the governing equations (45) and (47) are not "variationally consistent." However, the Galerkin approximation method may be applied to eqns (79) and (80) with no difficulty, resulting in a linear algebraic problem with a nonsymmetric coefficient matrix.

Now we consider the simplified case where the assumptions associated with Donnell's approximation are assumed to hold. Then eqns (45) and (47) are replaced by eqns (59) and (60). In this case, the weak form of the problem still consists of eqns (79) and (80), but with some modifications in the bilinear forms $A(\cdot,\cdot)$ and $D(\cdot,\cdot)$. More specifically, eqn (74) is replaced by

$$A(\bar{\phi},\phi)=\int_D\left(\bar{\phi}_{,ss}\beta_x\phi_{,ss}+\bar{\phi}_{,xx}\beta_s\phi_{,xx}+\bar{\phi}_{,ss}\frac{\rho}{2}\phi_{,xx}+\bar{\phi}_{,xx}\frac{\rho}{2}\phi_{,ss}\right)\mathrm{d}D \tag{81}$$

and eqn (77) is replaced by

$$D(\bar{\psi},\phi)=B(\bar{\psi},\phi). \tag{82}$$

We remark that in the *isotropic case* [see eqn (62)], the definition of the $A(\bar{\phi},\phi)$ in eqn (81) simplifies to

$$A(\bar{\phi},\phi)=\int_D(\nabla^2\bar{\phi})\frac{1}{Eh}(\nabla^2\phi)\,\mathrm{d}D. \tag{83}$$

Note that in the simplified case, the problem (79) and (80) is symmetric (self-adjoint) owing to eqn (82). Therefore, it is possible to recast it as the problem of finding stationary points of a functional. To this end, we define the functional $\mathscr{F}$, whose arguments are the functions $\phi\in\mathscr{S}_\phi$ and $\psi\in\mathscr{S}_\psi$:

$$\mathscr{F}[\phi,\psi]=\tfrac{1}{2}A(\phi,\phi)+\tfrac{1}{2}C(\psi,\psi)+B(\phi,\psi)-F(\psi). \tag{84}$$

Taking the first variation of $\mathscr{F}$ with respect to ϕ and to ψ and requiring that these variations vanish, yields eqns (79) and (80). As a consequence, the pair (ϕ,ψ) which makes $\mathscr{F}$ stationary is the solution of eqns (59), (60) and (65). In other words, eqns (59) and (60) are the Euler–Lagrange equations for the problem of finding the stationary points of the functional $\mathscr{F}$, and $\phi_{,xx}=0$ [see eqn (65)] is a natural boundary condition. From a mathematical standpoint, $\mathscr{F}$ is a Hellinger–Reissner type of integral; it has a single stationary point, which is a saddle point (Sewell, 1987).

9.2. *Second variational formulation*

The second variational formulation that can be used for the tube problem is an incremental variant of the mixed principle considered by Libai and Bert (1994), which is based on the principle of total potential energy. It is applicable only to the simplified problem (59) and (60) based on Donnell's approximation, and it is slightly more complicated than the variational formulation considered above. On the other hand, as we shall see in Section 9.3, it enables the use of more general boundary conditions, with clear physical interpretation. The latter reduce to those of small displacement theory when the incremental displacements and rotations are small.

The general form of the incremental functional is

$$\Pi^*[\phi, \psi] = \int_D (U^* + \bar{b}_{\alpha\beta} N^{\alpha\beta} \psi - \tfrac{1}{2} P^{\alpha\beta} \psi_{,\alpha} \psi_{,\beta} - p\psi)\, \mathrm{d}D, \tag{85}$$

where $U^* = \frac{1}{2}(N_x \varepsilon_x + N_s \varepsilon_s + N_t \gamma_{xs})$ is the incremental complementary energy (stress energy) measured from the reference state, $N^{\alpha\beta} = (N_x, N_s, N_t)$ are the incremental stress resultants, $\bar{b}_{\alpha\beta}$ are the curvatures in the reference configuration, and $P^{\alpha\beta} = (P_x, P_s, P_t)$ are the stress resultants in the reference state. All quantities in eqn (85) are expressed in terms of ϕ and ψ. In the present case, Π^* reduces to

$$\Pi^*[\phi, \psi] = \tfrac{1}{2} A^*(\phi, \phi) + \tfrac{1}{2} C(\psi, \psi) + B^*(\phi, \psi) - F(\psi), \tag{86}$$

where

$$A^*(\bar{\phi}, \phi) = \int_D (\bar{\phi}_{,ss} \beta_x \phi_{,ss} + \bar{\phi}_{,xx} \beta_s \phi_{,xx} + \bar{\phi}_{,xs} c_t \phi_{,xs} - \bar{\phi}_{,ss} \beta_{xs} \phi_{,xx} - \bar{\phi}_{,xx} \beta_{xs} \phi_{,ss})\, \mathrm{d}D \tag{87}$$

$$B^*(\bar{\phi}, \psi) = \int_D \bar{\phi}_{,xx} \frac{1}{R} \psi \, \mathrm{d}D \tag{88}$$

and where $C(\cdot, \cdot)$ and $F(\cdot)$ are as defined in eqns (76) and (78).

Taking the first variation of Π^* with respect to ϕ and to ψ and requiring that these variations vanish, yields eqns (79) and (80), with $A(\bar{\phi}, \phi)$, $B(\bar{\phi}, \psi)$ and $D(\bar{\psi}, \phi)$ replaced by $A^*(\bar{\phi}, \phi)$, $B^*(\bar{\phi}, \psi)$ and $B^*(\phi, \bar{\psi})$, respectively. As in the first variational formulation, it can be shown that the pair (ϕ, ψ) which makes Π^* stationary is also the solution of eqns (59), (60) and (65).

9.3. *Essential and natural boundary conditions*

It is also interesting to disregard the chosen boundary conditions (65), and to investigate *all* the essential and natural boundary conditions associated with the functionals $\mathscr{F}$ in eqn (84) and Π^* in eqn (85).

The boundary conditions on the edges $x = 0, L$ associated with the functional $\mathscr{F}$ in eqn (84) are:

$$\phi \text{ prescribed (essential)} \quad \text{or} \quad R\beta_s \phi_{,xxx} + R\frac{\rho}{2} \phi_{,ssx} + \psi_{,x} = 0 \text{ (natural)} \tag{89}$$

$$\phi_{,x} \text{ prescribed (essential)} \quad \text{or} \quad R\beta_s \phi_{,xx} + R\frac{\rho}{2} \phi_{,ss} = 0 \text{ (natural)} \tag{90}$$

$$\psi \text{ prescribed (essential)} \quad \text{or} \quad P_x \psi_{,x} + \frac{1}{R} \phi_{,x} = 0 \text{ (natural).} \tag{91}$$

We see that conditions (65) are indeed a subset of these boundary conditions; $\phi = 0$ and $\psi = 0$ are essential conditions, whereas $\phi_{,xx} = 0$ is the same as the natural condition in eqn (90), since $\phi = 0$ implies $\phi_{,ss} = 0$ on the edge. Similarly, the symmetry boundary conditions mentioned in Section 8.2 are also a subset of conditions (89)–(91). In the *isotropic* case [see eqn (62)], conditions (89) and (90) simplify to

$$\phi \text{ prescribed (essential)} \quad \text{or} \quad \frac{R}{Eh}(\nabla^2\phi)_{,x}+\psi_{,x}=0 \text{ (natural)} \tag{92}$$

$$\phi_{,x} \text{ prescribed (essential)} \quad \text{or} \quad \nabla^2\phi=0 \text{ (natural)}, \tag{93}$$

while conditions (91) remain unchanged.

The *essential* boundary conditions in eqns (89) and (90) have a simple physical interpretation; prescribing ϕ and $\phi_{,x}$ amounts to prescribing N_x and N_t, respectively [see eqn (57)]. In the small displacement case, ψ is equal to the normal displacement w (see Section 8), and so the essential boundary condition in eqn (91) means that w is prescribed. On the other hand, the *natural* boundary conditions in eqns (89)–(93) do not have any clear physical meaning, except as noted above.

The boundary conditions on the edges $x = 0$, L associated with the functional Π^* in eqn (85) are:

$$\phi \text{ prescribed (ess.)} \quad \text{or} \quad \beta_s\phi_{,xxx}-\beta_{xs}\phi_{,ssx}+c_t\phi_{,xss}+\frac{1}{R}\psi_{,x}=0 \text{ (nat.)} \tag{94}$$

$$\phi_{,x} \text{ prescribed (ess.)} \quad \text{or} \quad \beta_s\phi_{,xx}-\beta_{xs}\phi_{,ss}+\frac{1}{R}\psi=0 \text{ (nat.)} \tag{95}$$

$$\psi \text{ prescribed (ess.)} \quad \text{or} \quad P_x\psi_{,x}=0 \text{ (nat.)}. \tag{96}$$

The essential boundary conditions in this case are the same as in eqns (89)–(91). However, the natural boundary conditions in eqns (89)–(91) are different from those in eqns (94)–(96). The latter have clear physical interpretation, namely,

$$(94)\text{ (nat.)} \Rightarrow \varepsilon_{s,x}-\gamma_{xs,s}+\frac{1}{R}\psi_{,x}=0\,[\Rightarrow v_x=0] \tag{97}$$

$$(95)\text{ (nat.)} \Rightarrow \varepsilon_s+\frac{1}{R}\psi=0\,[\Rightarrow v_s=0] \tag{98}$$

$$(96)\text{ (nat.)} \Rightarrow P_x\psi_{,x}=0\,[\Rightarrow P_x w_{,x}=0]. \tag{99}$$

The conditions in square brackets are the small displacement conditions, which are obtained by substituting $\varepsilon_s = v_{s,s}-w/R$; $\psi = w$; $\gamma_{xs} = v_{s,x}+v_{x,s}$. The results for v_x and v_s agree with those of linear membrane shells, and that for $P_x w_x$ with that obtained for loaded shells. The second column in eqns (97)–(99) gives their intrinsic equivalents for the small-strain finite-displacement case. Note that the essential boundary conditions on ϕ and $\phi_{,x}$ in eqns (94) and (95) and the natural condition in eqn (96) are "equilibrium conditions," while the essential condition on ψ in eqn (96) and the natural conditions in eqns (94) and (95) [cf. eqns (97), (98)] are "kinematical conditions."

The natural condition (99) requires that either the edge is supported or the direction of the edge conforms with that of the applied edge force. The condition disappears when $P_x = 0$ and only two conditions remain. If a third condition is required (as in the case of a slitted tube), then it should be sought elsewhere (see Sections 8.3 and 8.4).

10. CONCLUDING REMARKS

In this paper we have presented a sequence of theories, with increasing order of simplification, for the nonlinear analysis of elastic membrane tubes. The approach was based on the linearization of the field equations about a known homogeneous reference state. This approach has led to a *linear* sixth-order system of equations, although the essential *nonlinear* effects of the membrane behavior were preserved. These equations are expressed via the stress function ϕ and the curvature function ψ.

We presented two variational formulations for this simplified problem. The first one is more general, in that it can be extended to deal with the problem governed by eqns (45) and (47). However, the second variational formulation includes more general boundary conditions, with clear physical interpretation. Each of the variational formulations may serve as a good basis for a classical Galerkin or finite element solution method. In fact, eqns (79) and (80) have the standard form of a mixed-type variational formulation which is often attacked by mixed finite element techniques. If the finite element method is employed, one may use rectangular elements with Hermite-cubic shape functions for ϕ and linear shape functions for ψ. Such an element would have 20 degrees of freedom [ϕ; $\phi_{,x}$; $\phi_{,y}$; $\phi_{,xy}$; ψ at each of the four nodes; see e.g. Johnson (1987)]. This is the simplest conforming element that can be used in conjunction with the proposed variational principles, and it involves significant computational effort. Therefore, in Part II of this series (Givoli and Libai, 1994) we shall use another solution technique. However, in doing this we shall have to restrict ourselves to tubes that do not contain slits or holes.

The problem of a small *longitudinal slit* in a membrane tube is a most interesting and difficult one. We have seen in Section 8.4 that this problem cannot be solved using standard linear membrane theory (as opposed to the case of a circumferential slit), and thus the approach proposed here seems most promising. We think that eqns (59) and (60) [or (61)] are a good starting point for an analytic or numerical analysis in this case. However, the presence of *singularity* at the tip of the slit adds a lot to the complexity of the problem, in two respects. First, one has to determine the correct form of singularity. Second, if a numerical scheme is employed, the singularity must be taken into account in the scheme. This may be done, for example, by appropriately refining the finite element mesh in the singularity region, but then elements other than rectangular ones must be used.

We have looked into the question regarding the nature of the singularity at the tip of the slit (in the isotropic case). We write $\phi \sim Ar^{\alpha}$ and $\psi \sim Br^{\beta}$ as $r \to 0$, where r is the distance from the tip, and we seek the values of the exponents α and β. From a purely mathematical standpoint, and assuming that we accept the variational formulation of Section 9.1 as our starting point, all the integrals appearing in eqns (74)–(77) must exist. This implies $\alpha > 1$ and $\beta > 0$. It is possible to pose the problem of finding α and β as an eigenvalue problem and to obtain an infinite sequence of eigenvalues. The smallest ones that satisfy $\alpha > 1$ and $\beta > 0$ are $\alpha = 3/2$ and $\beta = 1/2$. However, to establish that these indeed are the correct powers, one has to find an appropriate *physical* interpretation to the arguments made above.

Finally, we remark that the linearized equations obtained here, after linearizing the membrane equations about the known homogeneous state, may be viewed as the *leading* term in a perturbation series around the homogeneous state. Thus, it is appropriate to investigate the higher-order terms in this series in order to improve the approximation.

Acknowledgements—The second author acknowledges the support by the M. S. Geltman Memorial Academic Lectureship. Both authors acknowledge the support by the Fund for the Promotion of Research at the Technion.

REFERENCES

Budiansky, B. (1968). Notes on nonlinear shell theory. *ASME J. Appl. Mech.* **35,** 393.

Donnell, L. H. (1976). *Beams, Plates and Shells*. McGraw-Hill, New York.

Givoli, D. and Libai, A. (1995). Incremental stresses in loaded orthotropic circular membrane tubes—II. Numerical solution. *Int. J. Solids Structures* **32,** 1927–1947.

Gould, P. L. (1987). *Analysis of Shells and Plates*. Springer, Berlin.

Green, A. E. and Adkins, J. E. (1970). *Large Elastic Deformations*, 2nd edn. Clarendon Press, Oxford.

Green, A. E. and Zerna, W. (1968). *Theoretical Elasticity*, 2nd edn. Clarendon Press, Oxford.

Hughes, T. J. R. and Pister, K. S. (1978). Consistent linearization in mechanics of solids and structures. *Comput. Structures* **8,** 391.

Ibrahimbegovic, A. and Gruttmann, F. (1993). A consistent finite element formulation of nonlinear membrane shell theory with particular reference to elastic rubberlike material. *Finite Elements in Analysis and Design* **12,** 75.

Johnson, C. (1987). *Numerical Solutions of Partial Differential Equations by the Finite Element Method.* Cambridge University Press, Cambridge.

Koga, T. (1972). Bending rigidity of an inflated circular cylinder membrane of rubbery material. *AIAA J.* **10,** 1485.

Leonard, J. W. (1988). *Tension Structures.* McGraw-Hill, New York.

Libai, A. (1972). The nonlinear membrane shell with application to noncircular cylinders. *Int. J. Solids Structures* **8,** 923.

Libai, A. (1992). Nonlinear membrane theory. *Proc. Int. Cong. Theoretical and Applied Mechanics (ICTAM)* 1992, Haifa, Israel (Edited by Bodner *et al.*), p. 257. Elsevier, Amsterdam.

Libai, A. and Bert, C. W. (1994). A mixed variational principle and its application to the nonlinear bending problem of orthotropic tubes—I. Development of general theory and reduction to cylindrical shells. *Int. J. Solids Structures* **31,** 1003–1018.

Libai, A. and Simmonds, J. G. (1983). Nonlinear elastic shell theory. In *Advances in Applied Mechanics* (Edited by Hutchinson and Wu), Vol. 23, p. 271. Academic Press, Boston.

Libai, A. and Simmonds, J. G. (1988). *The Nonlinear Theory of Elastic Shells.* Academic Press, San Diego.

Needleman, A. (1976). Necking of pressurized spherical membranes. *J. Mech. Phys. Solids* **24,** 339.

Niordson, F. I. (1985). *Shell Theory.* North-Holland, Amsterdam.

Sewell, M. J. (1987). *Maximum and Minimum Principles.* Cambridge University Press, Cambridge.

Shield, R. T. (1972). On the stability of finitely deformed elastic membranes. *Z. Angew. Math. Phys.* **23,** 16.

Simo, J. C. and Pister, K. S. (1984). Remarks on rate constitutive equations for finite deformation problems: computational implications. *Comput. Meth. Appl. Mech. Engng* **46,** 201.

Steigmann, D. J. (1990). Tension field theory. *Proc. R. Soc. Lond.* **A429,** 141.